1. A: $f(x,y) = \begin{cases} \dfrac{x^2|y|}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

(for B relabel x and y)

(a) let $x = r\cos\theta,\ y = r\sin\theta$ (polar coordinates)

Then for $r > 0$,

$$f(x,y) = \frac{r^3\cos^2\theta|\sin\theta|}{r^2(\cos^2\theta+\sin^2\theta)} = r\cos^2\theta|\sin\theta|$$

$$\to 0 = f(0,0) \text{ as } r \to 0$$

Hence f is continuous at $(0,0)$

(b) let $u = (v,w) \in \mathbb{R}^2$ with $v^2+w^2=1$

For the directional derivation of f in the direction of u to exist, we need to consider $\dfrac{f(tv,tw)-f(0,0)}{t}$ for $t \to 0$

For $t \neq 0$, $$\frac{f(tv,tw)-f(0,0)}{t} = \frac{\frac{t^2|t|v^2|w|}{t^2(v^2+w^2)} - 0}{t}$$

$$= \frac{|t|}{t}v^2|w| = \operatorname{sgn}(t)\, v^2|w|.$$

which has a limit for $t \to 0$ only if $v^2|w| = 0$, i.e. $v=0$ or $w=0$.

(c) f is not differentiable at $(0,0)$.

One way to see this is that

if f would be differentiable at $(0,0)$

then $\nabla f(0,0) = (f_x(0,0), f_y(0,0)) = (0,0)$

where the partial derivatives exist (see (b))

Hence $D_u f(0,0) = \nabla f(0,0) \cdot u = 0$ f.a. unit vectors u

But in (b) it has been shown that

$D_u f(0,0)$ only exist for $u = (\pm 1, 0)$ or $u = (0, \pm 1)$.

Another way to see that f is not differentiable

at $(0,0)$ is to consider the linearization L

of f at $(0,0)$ and the definition of

differentiability.

$$L(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0)$$
$$= 0 + 0 \cdot (x-0) + 0 \cdot (y-0)$$
$$= 0$$

for $(x,y) \neq (0,0)$, $\dfrac{f(x,y) - L(x,y)}{\|(x,y)\|}$

$$= \frac{\frac{x^2|y|}{x^2+y^2} - 0}{\sqrt{x^2+y^2}} = \frac{x^2|y|}{(x^2+y^2)^{3/2}} = \cos^2\theta\,|\sin\theta|$$ which

↑ polar coordinates

has no limit for $r \to 0$

2. (a) By chain rule

$$w_s = f_x x_s + f_y y_s$$

$$w_{ss} = f_{xx} x_s^2 + f_{xy} y_s x_s + f_x x_{ss}$$
$$+ f_{yx} x_s y_s + f_{yy} y_s^2 + f_y y_{ss}$$
$$= f_{xx} x_s^2 + 2 f_{xy} x_s y_s + f_{yy} y_s^2$$
$$+ f_x x_{ss} + f_y y_{ss}$$

w_{tt} is obtained in the same manner (relabel)

(b) A: $x(s,t) = e^s \cos t$
$y(s,t) = e^s \sin t$

$\Rightarrow x_s = e^s \cos t, \; x_{ss} = e^s \cos t$
$x_t = -e^s \sin t, \; x_{tt} = -e^s \cos t$
$y_s = e^s \sin t, \; y_{ss} = e^s \sin t$
$y_t = e^s \cos t, \; y_{tt} = -e^s \sin t$

By part (a): $w_{ss} + w_{tt} =$
$f_{xx} e^{2s} \cos^2 t + 2 f_{xy} e^{2s} \cos t \sin t + f_{yy} e^{2s} \sin^2 t$
$+ f_x e^s \cos t + f_y e^s \sin t$
$+ f_{xx} e^{2s} \sin^2 t + 2 f_{xy} (-e^{2s} \cos t \sin t) + f_{yy} e^{2s} \cos^2 t$
$+ f_x (-e^s \cos t) + f_y (-e^s \sin t) = e^{2s}(f_{xx} + f_{yy})$
$= 0$ if $f_{xx} + f_{yy} = 0$

B: $x_s = \frac{t^2 - s^2}{(s^2+t^2)^2}$, $x_{ss} = 2s \frac{s^2-3t^2}{(s^2+t^2)^3}$

$x_t = -\frac{2st}{(s^2+t^2)^2}$, $x_{tt} = -2s \frac{s^2-3t^2}{(s^2+t^2)^3} = -x_{tt}$

$y_s = -x_t$, $y_{ss} = 2t \frac{t^2-3s^2}{(s^2+t^2)^3}$

$y_t = x_s$, $y_{tt} = -2t \frac{t^2-3s^2}{(s^2+t^2)^3} = -y_{ss}$

$w_{ss} + w_{tt} = \frac{1}{s^2+t^2} (f_{xx} + f_{yy})$

$= 0$ if $f_{xx} + f_{yy} = 0$

3. A

(a) let $F(x,y,z) = \frac{x^2}{9} + \frac{y^2}{4} + z^2$

$\Rightarrow S$ is level set of F for the value 3

$\nabla F(x,y,z) = \left(\frac{2}{9}x, \frac{1}{2}y, 2z\right)$

$\nabla F(3,-2,1) = \left(\frac{2}{3}, -1, 2\right)$

tangent plane

$\nabla F(3,-2,1) \cdot (x-3, y+2, z-1) = 0$

$\Leftrightarrow \frac{2}{3}(x-3) - (y+2) + 2(z-1) = 0$

$\Leftrightarrow \frac{2}{3}x - 2 - y - 2 + 2z - 2 = 0$

$\Leftrightarrow \frac{2}{3}x - y + 2z = 6$

$\Leftrightarrow 2x - 3y + 6z = 18$

(b) $\frac{\partial F}{\partial x}(x,y,z) = \frac{2}{9}x = \frac{2}{3} \neq 0$ for $(x,y,z) = (3,-2,1)$

By IFT $F(x,y,z) = 3$ is locally given

by the graph $x = f(y,z)$

$f_y = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial x}}$ and $f_z = -\frac{\frac{\partial F}{\partial z}}{\frac{\partial F}{\partial x}}$

$\Rightarrow f_y(y_0, z_0) = -\frac{\frac{1}{2}y_0}{\frac{2}{9}x_0} = -\frac{-1}{\frac{2}{9}\cdot 3} = \frac{3}{2}$

$f_z(y_0, z_0) = -\frac{2z_0}{\frac{2}{9}x_0} = -\frac{2}{\frac{2}{9}\cdot 3} = -3$

Linearization of f at (y_0, z_0):

$$L(y,z) = f(y_0, z_0) + f_y(y_0, z_0)(y - y_0) + f_z(y_0, z_0)(z - z_0)$$

$$= 3 + \frac{3}{2}(y+2) - 3(z-1)$$

$$= 9 + \frac{3}{2}y - 3z$$

$L(y,z) = x \iff x - \frac{3}{2}y + 3z = 9$

$\iff 2x - 3y + 6z = 18$

(c) find minimum of $f(x,y,z) = x^2 + y^2 + z^2$ under constraint $g(x,y,z) = 18$ where $g(x,y,z) = 2x - 3y + 6z$

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = 18 \end{cases} \iff \begin{cases} 2x = \lambda 2 \\ 2y = -\lambda 3 \\ 2z = \lambda 6 \\ 2x - 3y - 6z = 18 \end{cases}$$

$$\iff \begin{cases} x = \lambda \\ y = -\frac{3}{2}\lambda \\ z = 3\lambda \\ 2x - 3y - 6z = 18 \end{cases}$$

$\Rightarrow \quad 2\lambda + 3\frac{3}{2}\lambda - 6 \cdot 3\lambda = 18$

$$\lambda = -\frac{18}{14} = -\frac{9}{7}$$

$\Rightarrow \quad x = -\frac{9}{7}$

$$y = \frac{2}{3}\frac{9}{7} = \frac{6}{7}$$

$$z = -3 \cdot \frac{9}{7} = -\frac{27}{7}$$

4. A: $F(x,y,z) = ax\sin(\pi y)i + (x^2\cos(\pi y) + by e^{-z})j + y^2 e^{-z} k$

(a)

$$\nabla \times F(x,y,z) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ ax\sin(\pi y) & x^2\cos(\pi y) + bye^{-z} & y^2 e^{-z} \end{vmatrix}$$

$$= \left(2y e^{-z} + b y e^{-z}\right) i$$

$$- (0 - 0)\, j$$

$$+ \left(2x\cos(\pi y) - a\pi \cancel{x} \cos(\pi y)\right) k$$

$\nabla \times F(x,y,z) = 0$ f.a. $(x,y,z) \in \mathbb{R}^3$ requires

$b = -2$ and $a = \frac{2}{\pi}$

(b) let $F(x,y,z) = \frac{2}{\pi} x \sin(\pi y) i + (x^2\cos(\pi y) - 2ye^{-z}) j + y^2 e^{-z} k$

and ϕ be a potential function, i.e.

(*) $\frac{\partial \phi}{\partial x} = \frac{2}{\pi} x \sin(\pi y)$

(**) $\frac{\partial \phi}{\partial y} = x^2 \cos(\pi y) - 2y e^{-z}$

(***) $\frac{\partial \phi}{\partial z} = y^2 e^{-z}$

integrating (*) w.r.t. x gives

$$\phi(x,y,z) = \frac{1}{\pi} x^2 \sin(\pi y) + f(y,z)$$

Filling this in in (**) gives

$$\frac{\partial \phi}{\partial y} = x^2 \cos(\pi y) + f_y(y,z) = x^2\cos(\pi y) - 2ye^{-z}$$

$$\Rightarrow f_y(y,z) = -2ye^{-z}$$

$\Rightarrow \quad f(y,z) = -y^2 e^{-z} + g(z)$

$\Rightarrow \quad \phi(x,y,z) = \frac{1}{\pi} x^2 \sin(\pi y) - y^2 e^{-z} + g(z)$

Filling this in in $(***)$ gives

$$\frac{\partial \phi}{\partial z} = y^2 e^{-z} + g'(z) = y^2 e^{-z}$$

$$\Rightarrow g(z) = C \, , \quad C \in \mathbb{R}$$

$\Rightarrow \quad \phi(x,y,z) = \frac{1}{\pi} x^2 \sin(\pi y) - y^2 e^{-z} + C$

(c) By the Fundamental Th^m of Line Integrals the line integral is given by

$$\phi(1, \tfrac{1}{2}, 2) - \phi(0,0,0)$$

$$= \frac{1}{\pi} \sin\left(\frac{\pi}{2}\right) - \frac{1}{4} e^{-2} - 0$$

$$= \frac{1}{\pi} - \frac{1}{4e^2}$$

4. B: $F(x,y,z) = ax\cos(\pi y)\,i + (x^2\sin(\pi y) + bye^z)\,j + y^2e^z\,k$

(a) $\nabla \times F(x,y,z) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ ax\cos(\pi y) & x^2\sin(\pi y)+bye^z & y^2e^z \end{vmatrix}$

$$= (2ye^z - bye^z)\,i$$
$$-(0 - 0)\,j$$
$$+ (2x\sin(\pi y) + a\pi x\sin(\pi y))\,k$$

$\nabla \times F(x,y,z) = 0$ f.a. $(x,y,z) \in \mathbb{R}^3$ requires

$b = 2$ and $a = -\frac{2}{\pi}$

(b) let $F(x,y,z) = -\frac{2}{\pi}x\cos(\pi y)\,i + (x^2\sin(\pi y) + 2ye^z)\,j + y^2e^z\,k$

and ϕ be a potential function, i.e.

(*) $\frac{\partial\phi}{\partial x} = -\frac{2}{\pi}x\cos(\pi y)$

(**) $\frac{\partial\phi}{\partial y} = x^2\sin(\pi y) + 2ye^z$

(***) $\frac{\partial\phi}{\partial z} = y^2e^z$

integrating (*) w.r.t. x gives

$\phi(x,y,z) = -\frac{1}{\pi}x^2\cos(\pi y) + f(y,z)$

Filling this into (**) gives

$\frac{\partial\phi}{\partial y} = x^2\sin(\pi y) + f_y(y,z) = x^2\sin(\pi y) + 2ye^z$

$\Rightarrow f_y(x,y) = 2ye^z$

$\Rightarrow \quad f(y,z) = y^2 e^z + g(z)$

$\Rightarrow \quad \phi(x,y,z) = -\frac{1}{\pi} x^2 \cos(\pi y) + y^2 e^z + g(z)$

Filling this in in (**) gives

$$\frac{\partial \phi}{\partial z} = y^2 e^z + g'(z) = y^2 e^z$$

$$\Rightarrow g(z) = c, \quad c \in \mathbb{R}$$

$$\Rightarrow \phi(x,y,z) = -\frac{1}{\pi} x^2 \cos(\pi y) + y^2 e^z + c$$

(c): By the Fundamental Thm of Line Integrals the line integral is given by

$$\phi\left(1, \tfrac{1}{2}, 2\right) - \phi(0,0,0)$$

$$= -\frac{1}{\pi}\cos\left(\frac{\pi}{2}\right) + \frac{1}{4}e^2 - 0$$

$$= \frac{e^2}{4}$$

5.

A: $F(x,y,z) = (yz, -xz, z^3)$

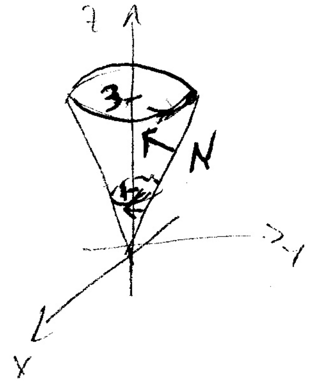
$$\nabla \times F(x,y,z) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ yz & -xz & z^3 \end{vmatrix}$$

$$= (0 - (-x))\, i - (0 - y)\, j + (-z - z)\, k$$

$$= x\, i + y\, j - 2z\, k$$

Let C be the cone oriented by

$$N = \nabla \left(z^2 - (x^2+y^2)\right) = (-2x, -2y, 2z)$$



C is bounded by the circles

S_1 : $x^2+y^2=1$ with clockwise orient.

and S_3 : $x^2+y^2=3^2$ with counter-clockwise orient.

By Stokes' Thm:

$$\iint_C \nabla\times F \cdot dS = \int_{S_1} F \cdot ds + \int_{S_3} F \cdot ds$$

$$\text{LHS}: \iint_{C_1'} \nabla\times F \cdot dS = \iint_{C_1'} (xi + yj - 2zk)\cdot \frac{N}{\|N\|}\, dS'$$

$$= \iint_{C_1'} \left(-2x^2 - 2y^2 - 4z^2\right) \frac{1}{\sqrt{4x^2+4y^2+4z^2}}\, dS'$$

$$= \iint_{C_1'} \left(-2z^2 - 4z^2\right) \frac{1}{\sqrt{4z^2+4z^2}}\, dS'$$

$$= \iint_{C_1'} \frac{-6z^2}{\sqrt{8}\, z}\, dS' = -\frac{3}{\sqrt{2}} \iint_{C_1'} z\, dS' =: (*)$$

parametrization of C_1':

$$\bar{X}(s,t) := (s\cos t, s\sin t, s), \quad s\in[1,3],\ t\in[0,2\pi]$$

$$\Rightarrow \frac{\partial \bar{X}}{\partial s} = (\cos t, \sin t, 1)$$

$$\frac{\partial \bar{X}}{\partial t} = (-s\sin t, s\cos t, 0)$$

$$\frac{\partial \bar{X}}{\partial s} \times \frac{\partial \bar{X}}{\partial t} = \begin{vmatrix} i & j & k \\ \cos t & \sin t & 1 \\ -s\sin t & s\cos t & 0 \end{vmatrix}$$

$$= -s\cos t\, i - s\sin t\, j + s\, k$$

$$\left\| \frac{\partial \bar{X}}{\partial s} \times \frac{\partial \bar{X}}{\partial t} \right\| = \sqrt{s^2+s^2} = \sqrt{2}\, s$$

$$\Rightarrow (*) = -\frac{3}{\sqrt{2}} \int_0^{2\pi}\int_1^3 s \cdot \sqrt{2}\, s\, ds\, dt$$

$$= -3\cdot 2\pi \int_1^3 s^2\, ds = -3\cdot 2\pi \cdot \frac{1}{3} s^3 \Big|_{s=1}^{s=3}$$

$$= -2\pi\left(3^3 - 1\right) = -52\pi$$

RHS: S_1 has paramtr: $r(t) = (\cos t, -\sin t, 1)$

$r'(t) = (-\sin t, \cos t, 0)$

$$\int_{S_1'} F \cdot ds = -\int_0^{2\pi} \underbrace{(\sin t, -\cos t, 1)}_{F(r(t))} \cdot \underbrace{(-\sin t, \cos t, 0)}_{r'(t)}\, dt$$

sign comes from orientation

$$= -\int_0^{2\pi} (-\sin^2 t - \cos^2 t)\, dt = 2\pi$$

S_3' has paramtr. $r(t) = (3\cos t, 3\sin t, 3)$

$r'(t) = (-3\sin t, 3\cos t, 0)$

$$\int_{S_3'} F \cdot ds = \int_0^{2\pi} \underbrace{(3^2 \sin t, -3^2 \cos t, 3)}_{F(r(t))} \cdot \underbrace{(-3\sin t, 3\cos t, 0)}_{r'(t)}\, dt$$

$$= \int_0^{2\pi} -3^3\, dt = -3^3 \cdot 2\pi.$$

$$\Rightarrow \int_{S_1'} F \cdot ds + \int_{S_3'} F \cdot ds = 2\pi(1 - 3^3) = -52\pi$$

6. $A: \vec{F}(x,y,z) = (x^2+y+z+z^2)i + (e^{x^2}+y^2)j + (3+x)k$

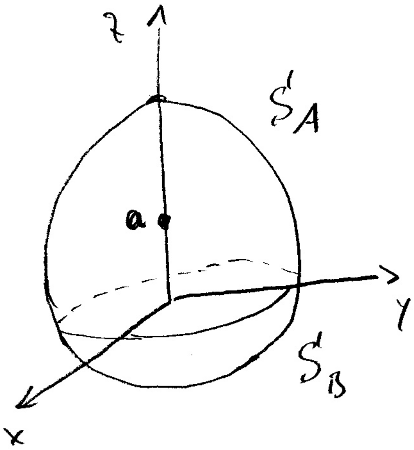
$B: \vec{F}(x,y,z) = (x^2+z+z+y^2)i + (e^{z^2}+y^2)j + (3+y)k$

Let S be the sphere $x^2+y^2+z^2 = 2az + 3a^2$

$\iff x^2+y^2+(z-a)^2 = 2a^2$

which we recognize as a sphere of radius $\sqrt{2}\,a$ centered at $(x,y,z) = (0,0,a)$.

Sketch:



S consists of the upper part S'_A where $z \geq 0$ and the lower part S'_B where $z \leq a$

where S'_A and S'_B are bounded by the circle which is the intersection of S' with the plane $z=0$. This circle satisfies $x^2+y^2 = 3a^2$, i.e. it is centered at the origin and has radius $\sqrt{3}\,a$.

Let D be the disk in the plane $z=0$ bounded by this circle. Orient D by $-k$ in case of A and k in case of B. Let $W_{A/B}$ be the solid enclosed by $S_{A/B}$ and D.

Then by Gauss' Divergence Th^{m}:

$$\iiint_{W_{A/B}} \nabla \cdot \vec{F} \, d\vec{V} = \iint_{\partial W_{A/B}} \vec{F} \cdot d\vec{S} = \iint_{S'_{A/B}} \vec{F} \cdot d\vec{S'} + \iint_{D} \vec{F} \cdot d\vec{S}$$

We have $\nabla \cdot \vec{F}(x,y,z) = 2x + 2y$.

Hence by symmetry $\iiint_{W_{A/B}} \nabla \cdot \vec{F} \, d\vec{V} = 0$.

So: $\iint_{S'_{A/B}} \vec{F} \cdot d\vec{S'} = - \iint_{D} \vec{F} \cdot d\vec{S}$

where D is oriented by $-k$ (A) or k (B)

$$-\iint_{D} \vec{F} \cdot d\vec{S} = - \iint_{D} \vec{F} \cdot (\mp k) \, dS' = \pm \iint_{D} (3+y) \, dS'$$

$= \pm \iint_{D} 3 \, dS$ (by symmetry)

$= \pm 3 \cdot$ area of D

$= \pm 3 \cdot \pi (\sqrt{3} a)^2$

$= \pm 9 \pi a^2$.

The outward flux through $S'_{A/B}$ is hence

$+/- \; 9 \pi a^2$.